# From Complexity to Intelligence

## Introduction to Inductive Reasoning and Proportional Analogy

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Pierre-Alexandre Murena

TELECOM ParisTech

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How do you define the Kolmogorov complexity of a string $\mathbf{x}$ ?

# Kolmogorov Complexity

How do you define the Kolmogorov complexity of a string **x** ?

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$$C_M(x) = \min_{p \in P_M} \{l(p); \quad p() = x\}$$

# Conditional Kolmogorov Complexity

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$$C_M(x|y) = \min_{p \in P_M} \{ l(p); \quad p(y) = x \}$$

# Minimum Description Length Principle

What is the MDL Principle ?

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## MDL Principle

The best theory to describe observed data is the one which minimizes the sum of the description length (in bits) of :

- the theory description
- the data encoded from the theory

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# Analysis of deduction

## Deduction examples (1)

1. All men are mortal.
2. Plato is a man.
3. Therefore, Plato is mortal.

## Cauchy-Schwarz inequality

Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ be two sequences of real numbers. Then :

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

## Proof

# Analysis of deduction
## Deduction examples (2)

### Cauchy-Schwarz inequality

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### Proof

For any $t \in \mathbb{R}$ :

$$0 \leq \|\alpha + t\beta\|^2 = \|\alpha\|^2 + 2\langle\alpha, \beta\rangle t + \|\beta\|^2 t^2 = P(t)$$

The quadratic polynomial $P$ is positive, so its discriminant is negative :

$$4|\langle\alpha, \beta\rangle|^2 - 4\|\alpha\|^2\|\beta\|^2 \leq 0$$

### Strong perfect graph theorem

A graph $G$ is perfect if for every induced subgraph $H$, the chromatic number of $H$ equals the size of the largest complete subgraph of $H$, and $G$ is Berge if no induced subgraph of $G$ is an odd cycle of length at least five or the complement of one.

### Proof

# Analysis of deduction

## Deduction examples (3)

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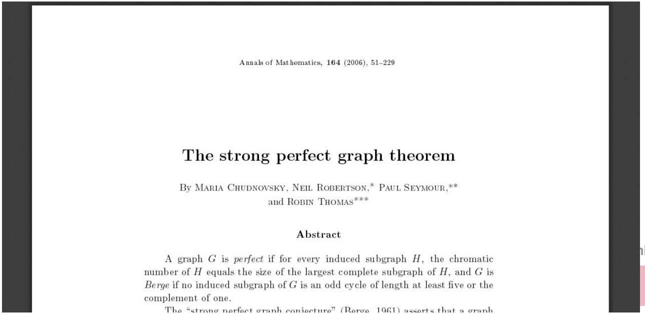
### Proof

**179 pages**



Annals of Mathematics, **164** (2006), 51–229

The strong perfect graph theorem

By Maria Chudnovsky, Neil Robertson,* Paul Seymour,** and Robin Thomas***

Abstract

A graph $G$ is *perfect* if for every induced subgraph $H$, the chromatic number of $H$ equals the size of the largest complete subgraph of $H$, and $G$ is *Berge* if no induced subgraph of $G$ is an odd cycle of length at least five or the complement of one.

The “strong perfect graph conjecture” (Berge, 1961) asserts that a graph

## A definition for deductive reasoning

Deductive reasoning is an approach where a set of logic rules are applied to general axioms in order to find (or more precisely *to infer*) conclusions of no greater generality than the premises.

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Deductive reasoning is an approach where a set of logic rules are applied to general axioms in order to find (or more precisely *to infer*) conclusions of no greater generality than the premises.

## Or, less formally :

- General $\longrightarrow$ Less general
- General $\longrightarrow$ Particular

# Limits of deduction

Will it rain today ?

# Limits of deduction

We are hardly able to get through one waking hour without facing some situation (e.g. *will it rain or won't it ?*) where **we do not have enough information** to permit deductive reasoning ; but still we must decide immediately.
In spite of its familiarity, the formation of plausible conclusions is a very subtle process.

in [Edwin T. Jaynes, *Probability theory. The logic of science*, Cambridge U. Press, 2003]

# Examples of conclusions of non-deductive reasoning

- It will rain today.
- All dogs bark.
- Everybody in this room knows that $1 + 1 = 2$
- The sun always rises in the East.
- Life is not a dream.
- $\ldots$

# Inductive reasoning

## Definition

Inductive reasoning is an approach in which the premises provide **a strong evidence** for the truth of the conclusion.

**The conclusion of induction is not guaranteed to be true !**

# A frequent confusion

**Deduction** : General rule $\Longrightarrow$ Particular case

**Induction** : Particular case $\Longrightarrow$ General rule

# A frequent confusion

**Deduction** : General rule $\Longrightarrow$ Particular case
**Induction** : Particular case $\Longrightarrow$ General rule

**This is incorrect !**

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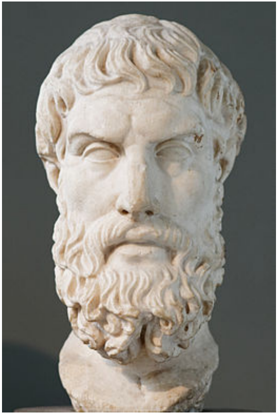
# Philosophical treatment

## Epicurus (342-270 B.C.)



**Principle of Multiple Explanations :** If more than one theory is consistent with the observations, keep all theories.

# Philosophical treatment

## Sextus Empiricus (160-210)

When they propose to establish the universal from the particulars by means of induction, they will effect this by a review of either all or some of the particulars. But if they review some, the induction will be insecure, since **some of the particulars omitted in the induction may contravene the universal** ; while if they are to review all, they will be **toiling at the impossible**, since the particulars are infinite and indefinite.

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1. **It is impossible to explore all possible situations.**
2. **How is it possible to know that the chosen individuals are representative of the concept ?**

# Philosophical treatment
## Example of a wrong induction

Do birds fly ?




# Philosophical treatment

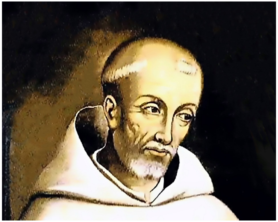
## Example of a wrong induction

Do birds fly ?


**No !**

# Philosophical treatment

## William of Ockham (1290-1349)



**Occam's Razor Principle :** Entities should not be multiplied beyond necessity

# Philosophical treatment

## Thomas Bayes (1702-1761)

Probabilistic point of view on inductive reasoning.

**Bayes's Rule :** The probability of hypothesis $H$ being true is proportional to the learner's initial belief in $H$ (the *prior probability*) multiplied by the conditional probability of $D$ given $H$.

# Philosophical treatment

## David Hume (1711-1766)

- Causal relations are not not found by deductive reasoning : just because a causal relation is stated in the past does not mean that it will be true in the future.
- Induction is based on a connection between the clauses "I have found that such an object has always been attended with such an effect" and I foresee that other objects which are in appearance similar will be attended with similar effects"
- Deduction cannot justify this connection ; but induction cannot justify it either.

# A fundamental question

**What is the justification for inductive reasoning ?**

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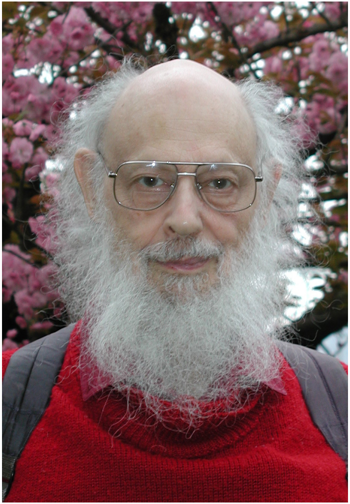
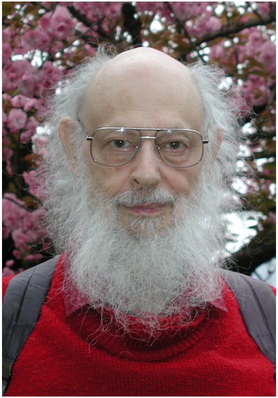
# Ray J. Solomonoff (1926-2009)



# General principle



Solomonoff's Lightsaber

Combining the **Principle of Multiple Explanations**, the **Principle of Occam's Razor**, **Bayes Rule**, using **Turing Machines** to represent hypotheses and **Algorithmic Information Theory** to calculate their probability.

## Principle of Multiple Explanations

All hypotheses explaining the data have to be considered.

Only the hypotheses discarded by the data can be rejected.

# Solomonoff's approach step by step

## Step 2 : Simplicity Principle

Even if all hypotheses are considered, the most complex hypotheses must be dropped when we find simpler ones.
This idea is basically derived from Occam's Razor.

# Solomonoff's approach step by step

## Step 3 : Bayes Rule

To neglect complex hypotheses, Bayes rule can be used with high priors for simple hypotheses and low priors for complex hypothes :

$$Pr(H_i|D) = \frac{Pr(D|H_i) \times Pr(H_i)}{Pr(D)}$$

where the value of $Pr(H_i)$ is low if $H_i$ is complex and high if $H_i$ is simple.



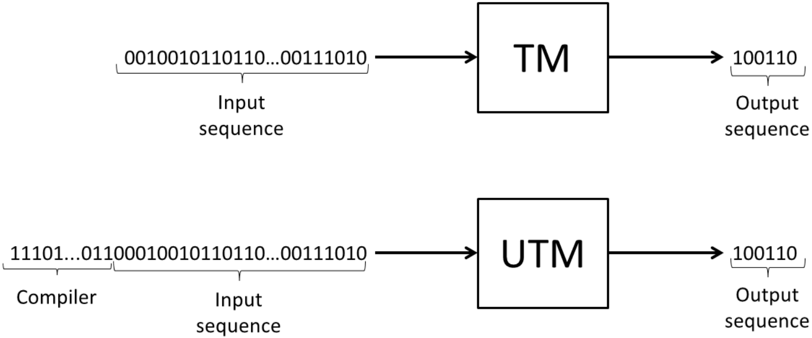
Pierre-Alexandre Murena

# Solomonoff's approach step by step
## Step 4 : Encoding hypotheses with Universal Turing Machines

- Data $D$ are encoded as a sequence over a finite alphabet $\mathcal{A}$ (for example binary alphabet $\mathcal{A} = \{0, 1\}$).
- Hypotheses are processes : hence, they can be represented as Turing Machines (TM).
- Hypotheses are represented as input sequences of Universal Turing Machines (UTM).
- The set of possible inputs of a UTM corresponds to the set of hypotheses.

# Solomonoff's approach step by step

## Step 4 : Encoding hypotheses with Universal Turing Machines



# Solomonoff's approach step by step
## Step 5 : Universal prior

The priors are chosen to be :

$$Pr(H_i) = 2^{-K(H_i)}$$

# Solomonoff's Induction

1. Run any possible hypothesis $H_i$ on the UTM :
   - If $H_i$ produces the data $D$ :
     1.1 Accept the hypothesis : $Pr(D|H_i) = 1$
     1.2 Calculate Kolmogorov complexity of $H_i$ : $K(H_i)$
     1.3 $Pr(H_i) = 2^{-K(H_i)}$
   - Otherwise : Discard the hypothesis : $Pr(D|H_i) = 0$
2. $H^* = \arg\max_{H_i}\{Pr(H_i) \times Pr(D|H_i)\}$

**This problem is intractable !**

## So what ?

The strongest result of this theory is that **a universal distribution can be used as an estimator *for all priors*.**

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### Theorem

If $\mu$ is the *concept* computable measure and the conditional semi-measure $\mu(y|x)$ is defined by $\mu(y|x) = \frac{\mu(xy)}{\mu(x)}$.
Let $\mathcal{B}$ be a finite alphabet and $x$ a word over $\mathcal{B}$. The summed expected squared error at the $n$-th prediction is defined by :

$$S_n = \sum_{a \in \mathcal{B}} \sum_{l(x)=n-1} \mu(x) \left( \sqrt{\mathbf{M}(a|x)} - \sqrt{\mu(a|x)} \right)^2$$

Then $\sum_n S_n \leq K(\mu) \log(2)$

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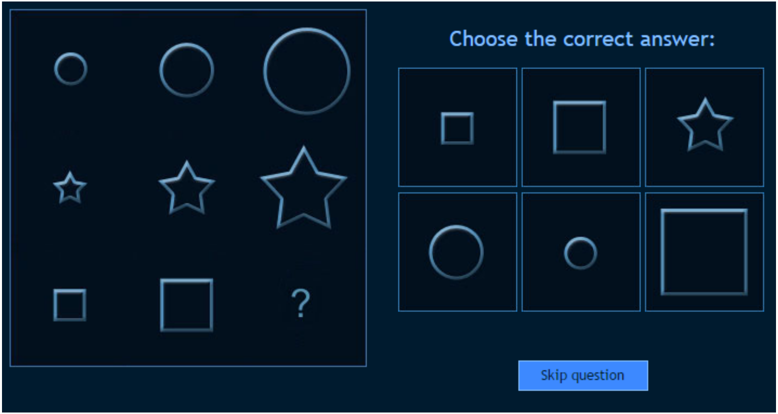
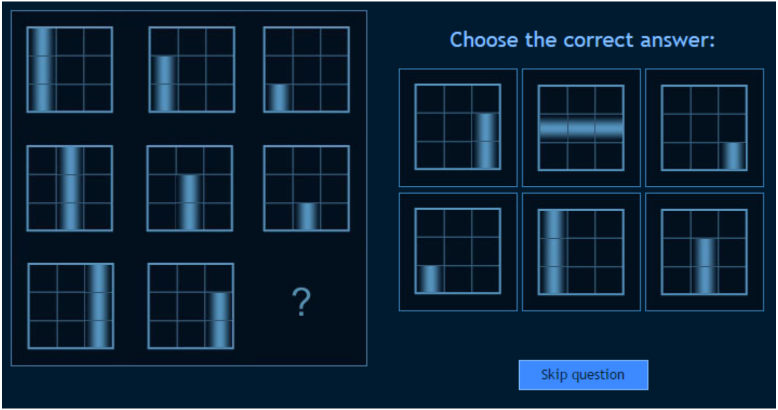
# IQ tests



# IQ tests



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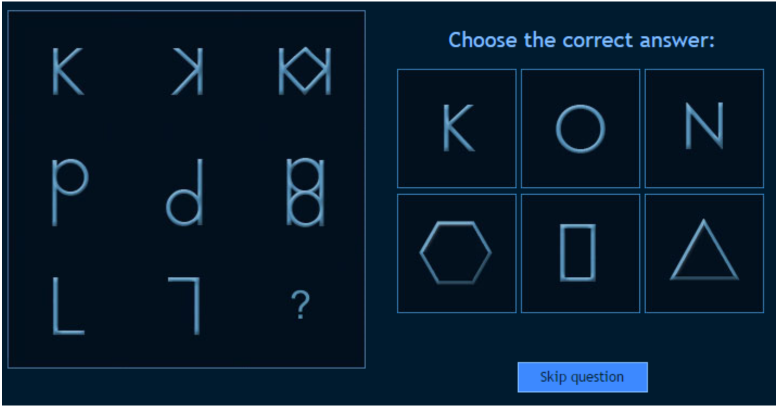

# What to say about these problems ?

- Inductive problems
- Repetition of *similar* structures
- A question is asked about a missing state
- Search of regularity

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**Such a situation is called an analogy**

# Analogy Reasoning

## Definition (Analogy reasoning)

Analogy reasoning is a form of reasoning in which one entity is inferred to be similar to another entity in a certain respect, on the basis of the known similarity between the entities in other respects.

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## Definition (Proportional Analogy)

Proportional Analogy concerns any situation of the form “A is to B as C is to D”

## Notation

$$A : B :: C : D$$

# Examples
## Analogy by Rendition

Occam's razor / Solomonoff's lightsaber

Works because of the underlying concept of *inductive principle*

# Examples

## Proportional analogy

- Gills are to fish as lungs are to man.
- François Hollande is to France as Vladimir Putin is to Russia
- Donald Trump is to Barack Obama as Barack Obama is to George Bush
- 37 is to 74 as 21 is to 42
- The sun is to Earth as the nucleus is to the electron

# Three axioms

The following axioms are commonly accepted (but not always) :

1. **Symmetry :** $A : B :: C : D \Leftrightarrow C : D :: A : B$
2. **Exchange :** $A : B :: C : D \Leftrightarrow A : C :: B : D$
3. **Determinism :** $A : A :: B : x \Rightarrow x = B$ and $A : B :: A : x \Rightarrow x = B$

# Analogy equation

**Definition (Analogy equation)**

$D$ is a solution of the analogy equation $A : B :: C : x$ iff $A : B :: C : D$

# Remarks on analogy equation

- Solving an analogy equation is a typical inductive reasoning problem.
- Several solutions may be equally correct for an equation
- The quality of a solution is dependent of the machine.

# Analogy algebra
## [Stroppa & Yvon, 2006]

Consider the division problem : $\frac{u}{v} = \frac{w}{x}$. This problem can be written as the problem of analogy $u : v :: w : x$

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The equation in $\mathbb{R}$ means that :

- $u = f_1 \times f_3$
- $v = f_1 \times f_4$
- $w = f_2 \times f_3$
- $x = f_2 \times f_4$

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This operation can be adapted to other domains.

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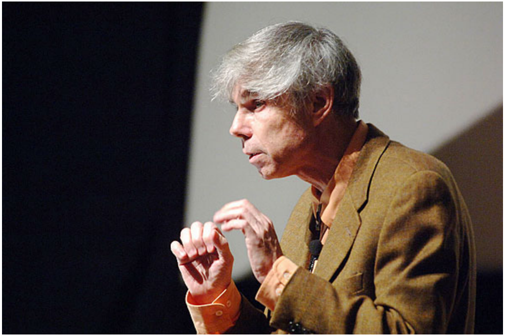
# Douglas Hofstadter (1945-now)



“*We are trying to put labels on things by mapping situations that we have encountered before. That to me is nothing but analogy.*”

# A micro-world

- Alphabet $\Sigma = \{A, B, C, \dots, Z\}$
- Elements of the analogy are words over $\Sigma$

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- Elements of the analogy are words over $\Sigma$

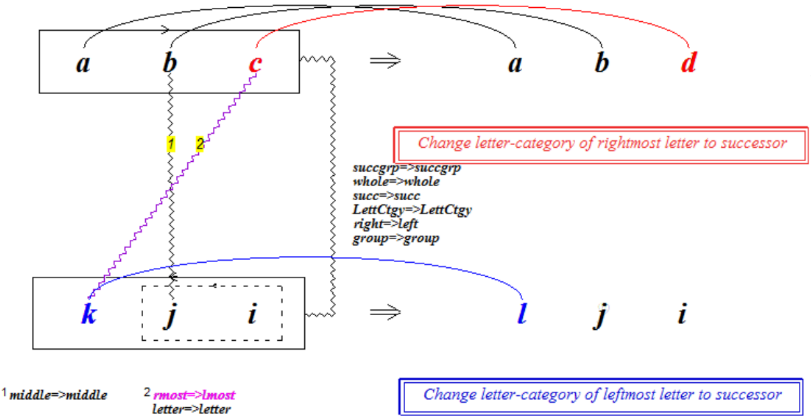
## Advantages of this micro-world

- Simplicity of the problems
- Human readibility
- Implies simple operations (predecessor, successor, add, remove, increment...)
- Covers a wide range of problems

# Problems you should know...

- ABC : ABD : : IJK : x
- RST : RSU : : RRSSTT : x
- ABC : ABD : : BCA : x
- ABC : ABD : : AABABC : x
- IJK : IJL : : IJJKKK : x
- ...

# An analogy solver : the Copycat project
## developed by Melanie Mitchell and Douglas Hofstadter



# An analogy solver : the Copycat project

**ABC : ABD : : IJK : : x**

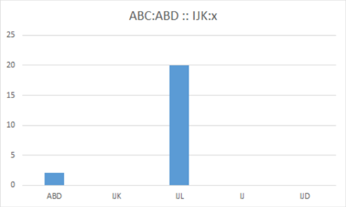
## Idea of Copycat

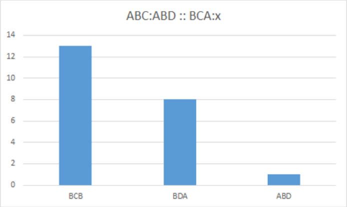
- Assembling *codelets* together to build up mappings between the strings
  - Mapping between source string **ABC** and target string **IJK**
  - Mapping between source string **ABC** and modified string **ABD**
- Identifying groups
- Building *bridges* supported by *concept-mapping*
- Building a short-term memory (the *slipnet*) to store concept mappings
- Creating a rule to describe the change of source string

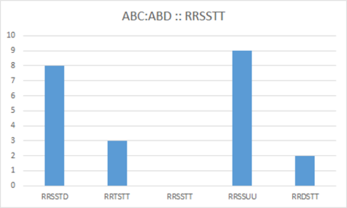
# Limitations of Copycat

- A very heuristic approach
- Lack of in-depth understanding of the found solutions
- Difficulties to solve simple problems : **A : A : : B : B**
- No memorization of the found answer

# Your results



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# Minimum Description Length Principle
... one more time ...

## MDL Principle

The best theory to describe observed data is the one which minimizes the sum of the description length (in bits) of :

- the theory description
- the data encoded from the theory

Let's try to apply the MDL Principle to analogy reasoning !

# Mathematical model

Consider the analogy equation $U : V :: W : x$

$$C(M) + C(D|M)$$

- $D$ correspond to the data : $D = \langle U, V, W \rangle$
- $M$ is a *global* model used to describe the data :
  - $M$ can be the description of the data
  - $M$ can be a description of a process generating data

**We propose to find assumptions to simplify the complexity term**

# Simplification of the MDL
## Separation of the models

### Hypothesis 1 : Separation of the models

The model $M$ is split in two parts : a source model $M_S$ and a target model $M_T$.

- $C(M) \leq C(M_S, M_T)$
- $C(D|M) = C(D|M_S, M_T)$

# Simplification of the MDL
## Transfer

### Hypothesis 2 : Model transfer

The target model is described with the help of the source model.

- $C(M) \leq C(M_S) + C(M_T|M_S)$
- $C(D|M) \leq C(D|M_S, M_T)$

# Simplification of the MDL

## Separation between source and target data

### Hypothesis 3 : Separation between source and target data

The source and target data are described with the help of their corresponding model only.

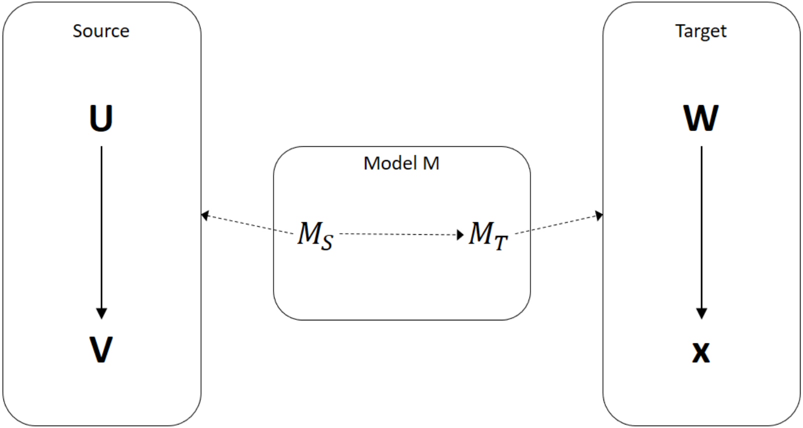
- $C(M) \leq C(M_S) + C(M_T|M_S)$
- $C(D|M) \leq C(D_S, D_T|M_S, M_T) = C(D_S|M_S) + C(D_T|M_T)$

### Important remark

The chosen simplification does not imply a transfer directly on the data, but on the models generating the data.

# Simplification of the MDL

## Summary



## And now ?

### Two approaches

- Find the $X$ minimizing $C(M_S) + C(U, V) + C(M_T|M_S) + C(W, x)$
- Find the target model minimizing

$$C(M_S) + C(U, V) + C(M_T|M_S) + C(W)$$

and infer $x$ from $M_T$ and $W$

# How to describe data with a model ?

## New assumptions

### Hypothesis 4 : Prevalence of inputs

Inputs are used to describe outputs.

- $C(M) \leq C(M_S) + C(M_T|M_S)$
- $C(D|M) \leq C(D_S|M_S) + C(D_T|M_T) \leq C(U|M_S) + C(V|M_S, U) + C(W|M_T) + C(x|M_T, W)$

# How to describe data with a model ?

## New assumptions

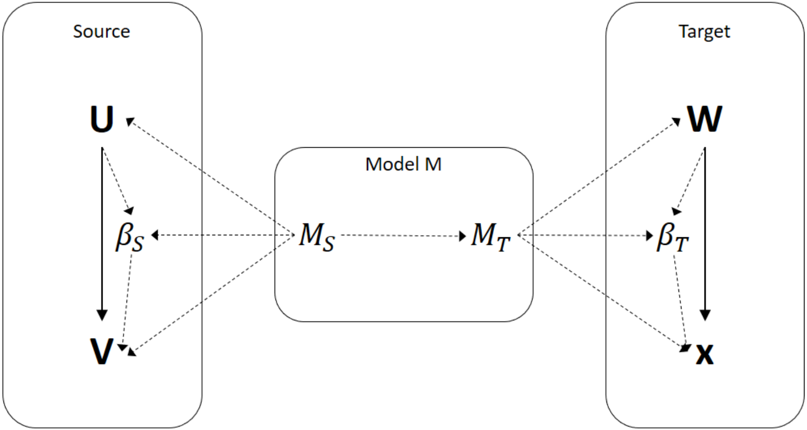
### Hypothesis 5 : Decision function

For both source and target, there exists a decision function (resp. $\beta_S$ and $\beta_T$).

- $C(M) \leq C(M_S) + C(M_T|M_S)$
- $C(V|M_S, U) \leq C(V, \beta_S|M_S, U) \leq C(\beta_S|M_S, U) + C(V|M_S, U, \beta_S)$
- $C(x|M_T, W) \leq C(\beta_T|M_T, W) + C(x|M_T, W, \beta_T)$

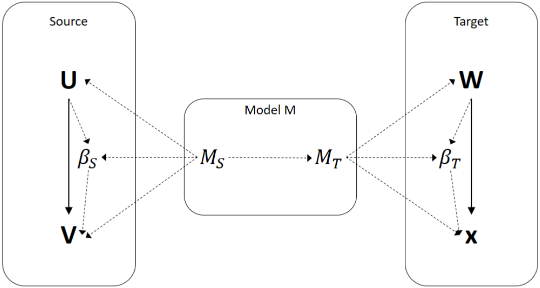
# Simplification of the MDL

## Summary



# Final equation

$$C(M_S) + C(U|M_S) + C(\beta_S|M_S, U) + C(V|M_S, U, \beta_S)$$
$$+ C(M_T|M_S) + C(W|M_S) + C(\beta_T|M_T, W) + C(x|M_T, W, \beta_T)$$



# Application : An example

Calculate **manually** the complexity of the proportional analogy :

**ABC : ABD : : IJK : x**

for the following values of **x** : IJL, ABD, IJK.

# Application : An example

Calculate **manually** the complexity of the proportional analogy :
**ABC : ABD : : IJK : x**
for the following values of **x** : IJL, ABD, IJK.

*Why not, but on which machine ?*

# Application : An example

## Choice of the UTM

- Orientation ($\rightarrow$ or $\leftarrow$) : 1 bit
- Cardinality $n$ : $\log(1+n)$ bits
- Length $l$ : $\log(1+l)$ bits
- Type : 3 bits

- A letter : 5 bits
  *Example* : C('g') = 5

- A string : C(orientation) + $\Sigma$ C(elements)
  *Example* : C('fci') = 1 + 3 $\times$ 5 = 16 bits

# Application : An example

## Choice of the UTM

- Ensemble : C(type of elements) + C(cardinality) + Σ C(elements)
  *Example* : C({ 'k', 'f', 'c' }) = 3 + 2 + 3 × 5 = 20 bits
- Group : C(type of elements) + C(number of elements) + Σ C(elements)
  *Example* : C({ 'u r l' }) = 3 + 2 + 3 × 5 = 20 bits
- Sequence : C(orientation) + C(type) + C(succession rule) + C(length) + C(first *or* last element)

# Application : An example
## Choice of the UTM

*Example* : length of the sequence 'abc'

- Orientation $\longrightarrow$ : C(orientation) = 1
- Type : letters : C(type) = 3
- Succession rule : function taking a letter as input (C(type=letter) = 3 bits) and taking its first successor (C(successor) = 1)
  Hence C(succession rule) = 4 bits
- Length 3 : C(length) = 2
- First element 'a' : C(first element) = 5 bits

**Hence C(sequence 'abc') = 1 + 3 + 4 + 2 + 5 = 15 bits**

# Application : An example

## The models

**ABC : ABD : : IJK : x**

- Model 1 : Generate a sequence of 3 letters and replace the third element by its successor (solution : IJL)
- Model 2 : Generate a sequence of 3 letters and replace the last element by its successor (solution : IJL)
- Model 3 : Return ABD (solution : ABD)
- Model 4 : Generate a sequence of 3 letters and change the ’c’ into a ’d’ (solution IJK)

# Application : An example

It’s your turn now !

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# Conclusion

## What to remember ?

- Difference between deduction and induction
- Non-universality of inductive reasoning
- Toward a universal solution : Solomoff's lightsaber
- What is analogy reasoning ?
- Using complexity to solve analogy equations ?

## What next ?

- Consider a large class of inductive problems : machine learning
- Apply MDL to machine learning problems

# Licence de droits d'usage

Contexte public } sans modifications