Emergence in Complex Systems

Introduction to Evolutionary Game Theory

Pierre-Alexandre Murena

TELECOM ParisTech

Information

These slides are not available on the main course page. You can download them (and the yesterday's presentation on Genetic Programming in Machine Learning) on my webpage:

`http://perso.telecom-paristech.fr/~pamurena`

Introduction

You already know that:

- Complex systems are systems made up of multiple agents
- The individual behaviors cannot be described.
- However, global behaviors **emerge** because of a propagation of the individual characteristics in evolution.

But you don't know any strict global mathematical explanation of this phenomenon! You don't even know if such a theory exists.

Such a framework exists: it is **Evolutionary Game Theory**.

Overview

1. **Introduction to Game Theory:** when two players are in a situation of simultaneous decision making, they can adopt optimal strategies (Nash equilibria).
2. **Evolutionary Stable Strategies:** when generalizing the classical games to populations, we notice that some equilibria are not stable when introducing different behaviors.
3. **Population dynamics:** the Evolutionary Stable Strategies are actually linked with mathematical properties of a very specific differential system (the replicator equation)

Outline

Outline

What is Game Theory?

- A definition by Roger B. Myerson: *Game Theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.*
- The tools developed by game theory can be applied in any situation where two (or more) agents have to take decisions which have impacts over all other agents (**interdependent actions**).

Applications of Game Theory

Game Theory has applications in various domains:

- Economics
- Biology
- Artificial Intelligence
- Resource allocation and Network management
- Politics

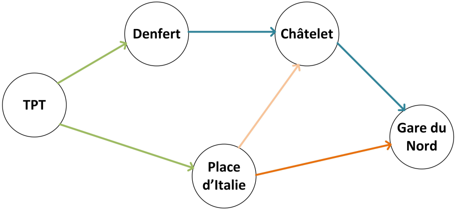
Elk fights
Biology

- Females are grouped in a harem held by a harem-holder male
- *N* other males want to access the harem
- A stranger entering the harem is chased away by the holder (fight)
- Males are wounded during the fight (depending on their strength)

Here, the individual strategy of each male depends on the *costs and gains* of a fight, but also on the *behavior of all other males*.

Congestion in public transports

Network management



- N students want to go from Télécom ParisTech to Gare du Nord
- The more people on the traject, the longer
- Three possible choices: (1) line 6 to Denfert then line B; (2) line 6 to Place d'Italie then line 5; (3) line 6 to Place d'Italie then line 7 to Châtelet then line B.

Pollution regulation

Politics / Economics

- N countries decide how much pollution they want to emit
- Each country n has a gain of $\beta_n(e_n)$ (benefits of emitting quantity e_n of pollution).
- Each country n has a deficit of $\phi_n(\sum_i e_i)$ because of the global pollution
- Which quantity will countries decide to emit?

Outline

Non-cooperative games and normal form

A game is called **non-cooperative** if the players make decisions independently. In a non-cooperative game, players cannot make coalitions or cooperation.

The **normal form** (also called **strategic form**) is the basic representation of a non-cooperative game.

Normal-form games

A normal-form consists in the description of:

- A set of players $I = \{1, \dots, n\}$
- A family of strategy sets $(S^i)_{i \in I}$
- A family of payoff functions $g^i : \prod_{j \in I} S^j \mapsto \mathbb{R}$

The play is made up of two parts:

1. All players $i \in I$ choose **simultaneously** a strategy $s^i \in S^i$
2. Each player $i \in I$ gets a reward $g^i(s^1, \dots, s^n)$

Remark: In the following, we will mainly consider *symmetric games* with two players, ie. games where the players have the same strategy sets: $S^1 = S^2 \triangleq S$

Examples

Coordination game

Two friends want to meet, either at place (A) or at place (B). They are satisfied only if they choose the same place

	A	B
A	$1,1$	$0,0$
B	$0,0$	$1,1$

Examples

Battle of the sexes

A married couple tries to decide what they will do this evening. The husband would rather watch football on TV (F) and the wife would rather go to the opera (O).

	F	O
F	$3, 1$	$0, 0$
O	$0, 0$	$1, 3$

Examples

Prisoner's dilemma

Two criminals are arrested by the police and are asked about their team. They have two choices: cooperate or defect.

	C	D
C	$-1, -1$	$-5, 0$
D	$0, -5$	$-3, -3$

Nash equilibrium in pure strategy

A Nash equilibrium in pure strategy is a couple of strategies $(p, q) \in S^2$ such that:

$$g^1(s, q) \leq g^1(p, q) \qquad \forall s \in S$$
$$g^2(p, s) \leq g^2(p, q) \qquad \forall s \in S$$

In other words, no player benefits from a change of strategy.
Not all games have a Nash equilibrium in pure strategy.

Examples

Battle of the sexes

	F	O
F	$3, 1$	$0, 0$
O	$0, 0$	$1, 3$

Two Nash equilibria in pure strategy: (F,F) and (O,O).
None of these equilibria corresponds to a payoff maximum for both players.

Examples

Rock-Paper-Scissor

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

There is no Nash equilibrium in pure strategy: for each strategy, at least one player may want to change its action.
What is the optimal way to play Rock-Paper-Scissor?

Mixed strategies and expected payoff

A **mixed strategy** is defined as a probability distribution over the pure strategies.

The set of mixed strategies is denoted $\Delta(S)$. For a finite number of strategies (of cardinal n), it corresponds to n-th dimensional simplex.

The expected payoff for player i corresponding to a couple of mixed strategies $(P, Q) \in \Delta(S)^2$ is defined as:

$$\mathbb{E}_{P,Q}[g^i(p,q)] = \sum_{p \in S} \sum_{q \in S} P(p)Q(q)g^i(p,q)$$

Mixed strategies and Nash equilibrium

Nash equilibrium in mixed strategy

A Nash equilibrium in mixed strategy is a couple of strategies $(P, Q) \in (\Delta(S))^2$ such that:

$$\begin{aligned} g^1(s, Q) &\leq g^1(P, Q) \qquad \forall s \in \Delta(S) \\ g^2(P, s) &\leq g^2(P, Q) \qquad \forall s \in \Delta(S) \end{aligned}$$

Nash theorem (1950)

For any finite game (finite number of players, finite number of strategies), there exists at least one Nash equilibrium in mixed strategy.

Example

Rock-Paper-Scissor

The strategy $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \Delta(S)$ defines a symmetric Nash equilibrium (ie. (p, p) is a Nash equilibrium) for the Rock-Paper-Scissor game.

The Hawk-Dove game

The players fight for a resource of value V. They can play two strategies:

- **Hawk:** trying to get the whole resource for himself
- **Dove:** not ready to fight for the resource

	H	D
H	$\frac{1}{2}(V-W), \frac{1}{2}(V-W)$	$V, 0$
D	$0, V$	$\frac{1}{2}V, \frac{1}{2}V$

The Hawk-Dove game

In the cas of $V < W$, there exists one single symmetric Nash equilibrium for the mixed strategy:

$$p = \left(\frac{V}{W}, 1 - \frac{V}{W} \right)$$

The game has also two pure Nash equilibria: (H, D) and (D, H) which are not biologically interesting (animals not labeled).

Outline

Outline

Players in population games

Biological games involve large populations of species in the games.

At each generation, two randomly picked individuals can meet and play the game $\mathcal{G}$ together. If $\sigma \in \Delta(S)$ is a mixed strategy, the two following perspectives are equivalent:

- Each individual plays a mixed strategy σ
- Individuals play only pure strategies. For each pure strategy $i \in S$, a proportion σ_i of the population plays pure strategy i.

A mixed strategy can be interpreted as a portion of the population playing a pure strategy.

Introducing mutants in a population

Consider a population of N individuals playing a game $\mathcal{G}$ (eg. Hawk-Dove). Suppose that this population is programmed to play a strategy x.

We now introduce a mutant population (in proportion ϵ) playing with the strategy y.

Question:

What condition on x guarantees that the base population won't be affected by mutants?

This condition will define **Evolutionary Stable Strategies** (ESS).

Population dynamics

- At generation t: proportion ϵ_t of mutants playing strategy y
- Mean strategy: $q_t = (1 - \epsilon_t)x + \epsilon_t y$
- Total fitness of the majority strategy: $F + g(x, q_t)$
- Total fitness of the mutant strategy: $F + g(y, q_t)$
- Assume the following dynamics:

$$\epsilon_{t+1} = \frac{F + g(y, q_t)}{F + g(q_t, q_t)}\epsilon_t \quad \Leftrightarrow \quad \epsilon_t - \epsilon_{t+1} = \frac{g(q_t, q_t) - g(y, q_t)}{F + g(q_t, q_t)}\epsilon_t$$

Stability of the majority strategy
Intuition

Intuition: the majority strategy is stable if the mutant population vanishes:

$$\epsilon_t \underset{t \longrightarrow \infty}{\longrightarrow} 0$$

We want $\epsilon_t - \epsilon_{t+1} > 0$ ie $g(q_t, q_t) > g(y, q_t)$ which can be expanded (knowing the development of q_t in terms of x and y):

$$(1 - \epsilon_t)[g(x, x) - g(y, x)] + \epsilon_t[g(x, y) - g(y, y)] > 0$$

Stability of the majority strategy

Three cases

$$(1-\epsilon_t)[g(x,x)-g(y,x)]+\epsilon_t[g(x,y)-g(y,y)]>0$$

- If $g(x,x)<g(y,x)$: LHS cannot be negative for small values of ϵ
- If $g(x,x)>g(y,x)$: LHS can be negative for small enough values of ϵ. The mutant population vanishes.
- If $g(x,x)=g(y,x)$: the mutant population vanishes if and only if $g(x,y)>g(y,y)$.

Outline

Definition of an ESS

Definition

An **Evolutionary Stable Strategy** of a symmetric two-person game $\mathcal{G} = \langle S, g \rangle$ is a strategy $x \in \Delta(S)$ satisfying the following conditions:

1. *Equilibrium condition:* (x, x) is a Nash equilibrium.
2. *Stability condition:* Every best reply y to x different from x satisfies $g(x, y) > g(y, y)$

Remarks:

- An ESS doesn't always exist for symmetric two-players games.

Examples

Prisoners dilemma

The unique Nash Equilibrium (D,D) is also an ESS.

Harm thy neighbor

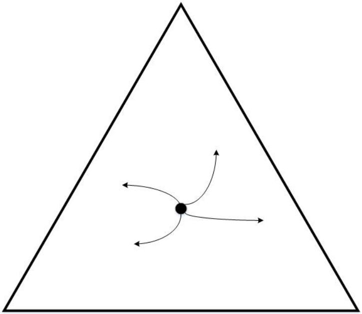
	A	B
A	$2, 2$	$1, 2$
B	$2, 1$	$2, 2$

(A,A) and (B,B) are both Nash equilibria, but only (B,B) is an ESS.

Examples

Rock-Paper-Scissor

The only Nash equilibrium $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not an ESS.
You can prove it with the definition of the ESS, but would you be able to explain it?



Outline

Outline

Description of the replicator model

- Consider a population of N individuals (called replicators)
- Each individual plays only one pure strategy
- Each individual passes its strategy to its descendants
- The number of descendants depends linearly of the mean gain of the parent (called fitness)
- The standard birth (resp. death) ratio is β (resp. δ)
- The number of individuals playing i-th strategy is denoted by $n_i = x_i N$

Population evolution

- Replicators playing i-th strategy:

$$\dot{n}_i = (\beta - \delta + g(s_i, \mathbf{x}))\, n_i$$

- Global population:

$$\dot{N} = \sum_{i=1}^{n} \dot{n}_i = (\beta - \delta) \underbrace{\sum_{i=1}^{n} n_i}_{=N} + \underbrace{\left(\sum_{i=1}^{n} x_i g(s_i, \mathbf{x}) \right)}_{=g(\mathbf{x},\mathbf{x})} N$$

- Link between individuals and global population:

$$\dot{n}_i = N \dot{x}_i + x_i \dot{N}$$

Replicator equation

From the previous results, we get:

$$N\dot{x}_i = (\beta - \delta + g(s_i, \mathbf{x}))x_i N - x_i(\beta - \delta + g(\mathbf{x}, \mathbf{x}))N$$

And finally:

Replicator equations

$$\boxed{\dot{x}_i = \left(g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x})\right) x_i}$$

Outline

Fixed points in replicator dynamics

Definition: Fixed point

A **fixed point** of the replicator dynamics is a strategy that satisfies $\dot{x}_i = 0$ for all pure strategy i.

Definition: Asymptotically stable fixed point

A fixed point $\mathbf{x}^*$ of the replicator dynamics is called **asymptotically stable** if for each pure strategy $i \in S$ there exists $\delta > 0$ such that $|x_i(0) - x_i^*| < \delta$ implies $\lim_{t \longrightarrow \infty} |x_i(t) - x_i^*| = 0$.

Fixed points and Nash equilibrium

Theorem

If $\sigma \in \Delta(S)$ is a mixed strategy of $\mathcal{G}$ such that (σ, σ) is a symmetric Nash equilibrium, then the state $\mathbf{x} = \sigma$ is a fixed point of the replicator equation.

Theorem

If $\mathbf{x}$ is an asymptotically stable fixed point of the replicator equation and $\sigma \in \Delta(S)$ is the mixed strategy of $\mathcal{G}$ associated to the state $\mathbf{x}$, then the symmetric strategy (σ, σ) is a symmetric Nash equilibrium.

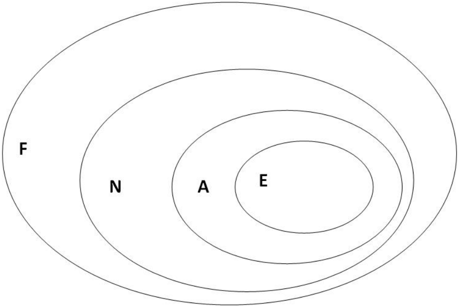
Theorem

If $\sigma \in \Delta(S)$ is a mixed strategy of the game $\mathcal{G}$ such that (σ, σ) is an ESS of $\mathcal{G}$, then the state $\mathbf{x}$ associated to σ is an asymptotically stable fixed point of the corresponding replicator equation.

Summary of the theorems

If $\mathcal{G}$ is a game, we consider the following sets:

- **F**: fixed points of the replicator equation
- **N**: symmetric Nash equilibrium of $\mathcal{G}$
- **A**: asymptotically stable points of the replicator equation
- **E**: ESS of $\mathcal{G}$



Outline

Replicator for two-strategies games

Suppose that the played game $\mathcal{G}$ has only two strategies. We can choose the notations $x_1 = x$ and $x_2 = 1 - x$.
The replicator equation can be rewritten as:

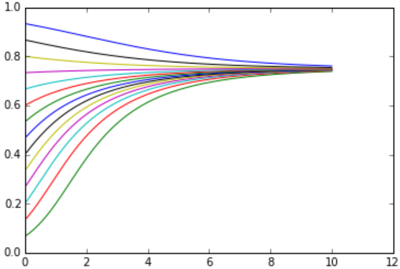
$$\begin{aligned}\dot{x} &= (g(s_1, x) - g(\mathbf{x}, \mathbf{x}))\, x \\ &= (g(s_1, \mathbf{x}) - xg(s_1, \mathbf{x}) - (1 - x)g(s_2, \mathbf{x}))\, x \\ &= x(1 - x)[g(s_1, \mathbf{x}) - g(s_2, \mathbf{x})]\end{aligned}$$

Hawk-Dove

Replicator equation:

$$\dot{x} = \frac{1}{2}x(1-x)(V-Wx)$$

Solutions of the replicator equation for $V = 3$ and $W = 4$:

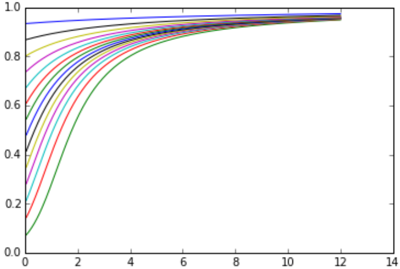


Hawk-Dove

Replicator equation:

$$\dot{x} = \frac{1}{2}x(1-x)(V-Wx)$$

Solutions of the replicator equation for $V=4$ and $W=4$:



Prisoner's dilemma

	C	D
C	α, α	$0, \beta$
D	$\beta, 0$	γ, γ

Replicator equation:

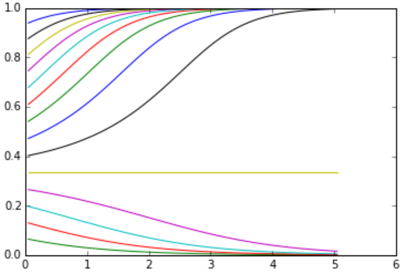
$$\dot{x} = x(1 - x)[(\alpha - \beta + \gamma)x - \gamma]$$

Prisoner's dilemma

Replicator equation:

$$\dot{x} = x(1 - x)[(\alpha - \beta + \gamma)x - \gamma]$$

Solutions of the replicator equation for $\alpha = 5, \beta = 3$ and $\gamma = 1$:



Rock-Paper-Scissor

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

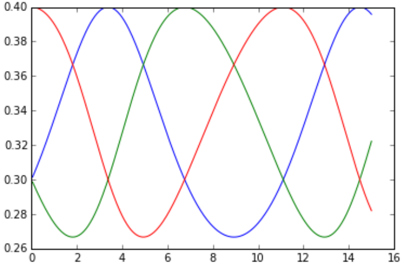
$$g(s_1, \mathbf{x}) = x_3 - x_2$$
$$g(s_1, \mathbf{x}) = x_3 - x_2$$
$$g(s_1, \mathbf{x}) = x_3 - x_2$$
$$g(\mathbf{x}, \mathbf{x}) = x_1(x_3 - x_2) + x_2(x_1 - x_3) + x_3(x_2 - x_1)$$

Rock-Paper-Scissor

Replicator equation:

$$\dot{x}_i = [g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x})]x_i$$

Solutions of the replicator equation with initial state $x = \frac{1}{10}(3, 3, 4)$:

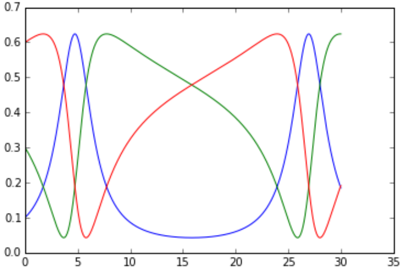


Rock-Paper-Scissor

Replicator equation:

$$\dot{x}_i = [g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x})]x_i$$

Solutions of the replicator equation with initial state $x = \frac{1}{10}(1, 3, 6)$:



Outline

Conclusion

- Evolutionary population effects can be studied with the perspective of game theory
- Traditional game theory introduces the notion of *Nash equilibrium* which describe compromise in social interactions
- Evolutionary game theory describes the convergence to Nash equilibria
- Emergence phenomenons in complex systems are not magical effects: they are the result of a convergence to a Nash equilibrium
- Equilibrium points exist in all systems, but they cannot necessarily emerge

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